Theoretical Design of Shock-Free, Transonic Flow around Aerofoil Sections

G. Y. NIEUWLAND

National Aerospace Laboratory N.L.R., Amsterdam

SUMMARY

A new development of the hodograph theory of Chaplygin, Cherry and Lighthill is based on integral transform methods. Transonic potential flows around a family of both non-lifting and lifting quasi-elliptical aerofoils are obtained. Various examples are shown, and the use of the solutions in the design of shock-free transonic flows is discussed.

1. INTRODUCTION

The use of function theoretic methods in the theory of plane compressible fluid flow, based on the hodograph transformation, has a long history going well back into the nineteenth century. By the hodograph transformation, introduced in fluid dynamics by the Dutch mathematician Molenbroek in in 1890[1], the quasi-linear partial differential equation for the plane potential flow of a gas is transformed into a linear one by a change of dependent and independent variables. This permits the study of non-linear flow phenomena by the solution of linear problems.

The first practical application was given by Chaplygin (1904), in his classical paper[2] on the subsonic outflow of a gas jet from a slit. His basic idea was to construct a series expansion for the analytic potential describing the incompressible jet flow in hodograph variables, and then to substitute particular solutions of the hodograph equation for compressible flow (the Chaplygin particular solutions) into this series, thus relating a compressible flow to a given incompressible one.

The first application of hodograph methods to the problem of compressible flow around a body is given by the well-known von Karman–Tsien theory[3]

(1939), in which a particular simplifying approximation to the pressure-density relation is used; this results however in these flows remaining entirely subsonic.

The theory obtained its definitive form mainly in the hands of Bergman, Cherry and Lighthill. The first solution for a transonic flow related to the non-circulatory incompressible flow around a circular cylinder was given by Goldstein, Lighthill and Craggs[4], and independently by Cherry[5], who also presented a numerically worked out example. This work was later extended to the circulatory flow case by Lighthill[6], and by Cherry[7]. Lighthill also gave a more general integral operator form of the theory[8]; a basically similar theory also had been developed by Bergman[9] on the basis of his general theory of integral operators. This form of the theory is applicable to any incompressible potential flow given in the physical plane, but restricted to subsonic flow speeds. These solutions can be extended into the transonic régime when series expansions of a certain type for the hodograph potential of the original flow are available. This condition means, however, a severe restriction on the class of admissible incompressible flows.

This general class of solutions, when defining transonic potential flows, was the subject of an at times rather emotional debate on what became known as the 'transonic controversy'. These discussions resulted from attempts to give both mathematical and physical interpretations to these results.

At first, the controversy took the simple form, that the mathematical solutions seemed to indicate the possibility of continuous transition from subsonic to supersonic flow speeds and back, a phenomenon generally not observed in wind tunnel experiments. Experimentally, under transonic flow conditions recompression of the flow from locally supersonic regions to subsonic flow speeds usually occurs through shock waves, this often being accompanied by the occurrence of shock-induced boundary layer separation, so that any basis for comparison with a potential flow solution is lost.

It was then suggested by Busemann and Guderley[10] that transonic potential flow solutions lacked any physical significance whatsoever, as these could be interpreted as solutions of a boundary value problem that is not continuously dependent on its boundary data. The latter fact, conjectured from physical arguments and made plausible by constructing particular examples by these authors, was later proved in a mathematically rigorous fashion in a series of fundamental papers by Morawetz (sec. 4). The Busemann–Guderley argument, so forcefully underlined by the intricacies of experimentally observed transonic flows, became more or less generally accepted and tended to lessen the practical aerodynamicist's interest in the class of solutions under discussion.

At this point, new developments were brought in from the experimental side. At the National Physical Laboratory, Holder and Pearcey were engaged in an experimental programme on the design of wing sections for swept-back wings in the transonic speed range[11]. They started from the idea, that in transonic flow control of the boundary layer development is of prime importance, and that especially the shock-induced boundary layer separation should be suppressed. This led to attempts to reduce the shock strength by a special design of the aerofoil shape, which have been described by Pearcey at the Zürich I.C.A.S. Congress[12]. In some cases, Pearcey succeeded in virtually eliminating the shock wave,† obtaining a to all intents and purposes shock-free transonic flow with a local supersonic region of appreciable extent and a high local maximum Mach number. Pearcey pointed out that the aimed at sections with a 'peaky pressure distribution' are of considerable interest to the aircraft designer from the point of view of reducing transonic drag rise. However, these flows up to the present time are obtained on a largely empirical basis.

These new developments then, clearly justify a renewed interest in the classical hodograph methods. On the one hand the transonic potential flow solutions become important as a theoretical reference for experimental studies on the now proven feasibility of transonic shock-free flow, and generally as a basis for design methods for these flows. On the other hand, Morawetz's theorems indicate that it would be at least extremely difficult to obtain precise solutions in any other way; in particular these cannot be obtained by posing a boundary value problem.

Moreover, by now developments in both numerical methods and computer technology make it possible to devise computational methods to construct flow patterns in the range of practical interest, which was impossible at the time the mathematical theories were developed. As will be demonstrated, it has in fact been possible to construct theoretically transonic potential flow patterns that have the general characteristics of the flows around aerofoil sections shown by Pearcey to be conducive to a shock-free real flow.

At the NLR, Amsterdam, we have for several years now been working on the computation of these flows. In particular, a theory of quasi-elliptical aerofoils has been developed, i.e. compressible flows related to the incompressible flow around an elliptical cylinder, both in the zero lift and in the lifting case. The, unfortunately, rather involved analysis required for the circulatory flow problem gave occasion to reconsider the basic structure of the theory, and we developed a new version based on integral transform techniques.

Now this, as a merely technical device does, of course, not essentially extend the theory, which is still firmly based in particular on Lighthill's work. On the other hand, this new interpretation immediately clarifies the idea of 'mapping' analytic potentials into the solution space of the Chaplygin equation, and unifies and generally tidies up the existing theory.

In the following a brief sketch of this theory will be attempted, referring

† He has promised to present some experimental evidence in the Discussion.

for the mathematical details to Lighthill's papers[8], and to ref. 13, to be issued shortly. Then some examples of our results will be shown, leaving aside the most difficult practical problem: the development of the appropriate numerical techniques. At the end, we will return to what is perhaps the most intriguing question of all — the physical interpretation, if any, of these theoretical results.

However, before going on I would like to acknowledge the work of my colleagues W. J. Boerstoel, G. van Eek and M. J. M. G. van Gennip, who contributed significantly in various parts of this project, in particular in the crucial numerical work.

List of Symbols

a, b	real numbers, p. 215
a_ν, b_ν	eq. (5)
C_m	eq. (6)
$f_\nu(\tau_1)$	compressibility factor, eqs. (13), (16), (17)
I_0	analytic function, eq. (18*a*)
k	real number, p. 215
L, L^*	hodograph series, eqs. (22), (23)
M	Mach number
$M(\alpha, \beta; a, b)$	class of analytic functions, p. 215
q	flow velocity
$q_{\max}$	thermodynamically maximum flow speed
$s(\tau)$	velocity function, eq. (2)
$t(\tau)$	velocity function, eq. (7*c*)
$T(\tau)$	eq. (3)
$V(\tau)$	eq. (7*a*)
$z = x + iy$	physical plane complex co-ordinate
α, β	angles, p. 215
γ	specific heat ratio
Γ	circulation $\times (1/2\pi)$
ε	ellipticity factor
ζ	conjugate velocity vector
ζ_1, ζ_2	branch points in hodograph, eq. (10)
θ	flow angle
ρ	flow density
σ	eq. (2*a*)
τ	velocity parameter $(q/q_{\max})^2$
Φ	complex potential
ψ	stream function
$\psi_\nu(\tau)$	Chaplygin function, eq. (5)

2. The Analytical Hodograph Theory

2.1 *The hodograph equation; Chaplygin particular solutions*

The basis of the hodograph theory is the Chaplygin equation for the stream function of the plane potential flow of a compressible medium obeying the isentropic gas law:

$$\tau(1-\tau)\psi_{\tau\tau}+\left(1+\frac{2-\gamma}{\gamma-1}\tau\right)\psi_{\tau}+\frac{1}{4\tau}\left(1-\frac{\gamma+1}{\gamma-1}\tau\right)\psi_{\theta\theta}=0 \tag{1}$$

where

$$\tau=\left(\frac{q}{q_{\max}}\right)^{2}, \quad 0\leqq\tau<1$$

$$=\frac{1}{1+\dfrac{2}{\gamma-1}\dfrac{1}{M^{2}}}$$

The compressibility in the flow is governed by the magnitude of the thermodynamically maximum possible speed $q_{\max}$ with respect to a fixed reference speed, for which the asymptotic velocity q_{∞} in the physical plane is chosen. Defining $q_{\infty}=1$, we write

$$\tau_{1}=\left(\frac{1}{q_{\max}}\right)^{2}$$

Introducing the variable $s(\tau)$ by

$$\frac{\mathrm{d}s}{\mathrm{d}\tau}=\frac{1}{2\tau}(1-M^{2})^{1/2} \tag{2}$$

one can bring eq. (1) in its subsonic normal form:

$$\psi_{ss}+\psi_{\theta\theta}=T(s)\psi_{s} \tag{3}$$

$$T(s)=\frac{\mathrm{d}}{\mathrm{d}s}\log\{\rho(1-M^{2})^{-1/2}\}$$

$$\rho=(1-\tau)^{1/(\gamma-1)}$$

The definition of s involves an integration constant σ, which can be fixed by the condition:

$$\mathrm{e}^{s}\rightarrow\tau^{1/2} \qquad \text{for} \qquad q_{\max}\rightarrow\infty \tag{4}$$

σ then becomes the value of s for the sonic speed $[\tau = (\gamma-1)/(\gamma+1)]$:

$$\sigma = -\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tanh^{-1}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} + \tfrac{1}{2}\log[2(\gamma-1)] \qquad (2a)$$

The Chaplygin particular solutions of eq. (1) are

$$\psi_\nu(\tau)\,\mathrm{e}^{\pm i\nu\theta}, \qquad (5)$$

$$\psi_\nu(\tau) = \tau^{\nu/2} F(a_\nu, b_\nu; \nu+1; \tau)$$

$$a_\nu + b_\nu = \nu - \frac{1}{\gamma-1}$$

$$a_\nu b_\nu = -\frac{\nu(\nu+1)}{2(\gamma-1)}$$

and F denotes the (ordinary) hypergeometric function.

The Chaplygin function $\psi_\nu(\tau)$ is a meromorphic function of the complex variable ν, with simple poles at $\nu = -2, -3, \ldots, -m, \ldots$ having residues which can be written:

$$-mC_m\psi_m(\tau) \qquad (6)$$

$$C_m = \frac{(a_m-1)!\,(m-b_m)!}{(a_m-m-1)!\,(-b_m)!\,(m!)^2}$$

$$\sim \frac{1}{2\pi m}\mathrm{e}^{-2\sigma m} \quad \text{for large } m$$

For subsonic speeds $[\tau < (\gamma-1)/(\gamma+1)]$, $\psi_\nu(\tau)$ behaves asymptotically for large ν as

$$\psi_\nu(\tau) \sim V(\tau)\,\mathrm{e}^{\nu s} \qquad (7a)$$

$$V(\tau) = \left\{\frac{(1-\tau)^{(\gamma+1)/(\gamma-1)}}{1-[(\gamma+1)/(\gamma-1)]\tau}\right\}^{1/4};$$

for supersonic speeds there is a complicated oscillatory behaviour:

$$\psi_\nu(\tau) \sim 2\,|V(\tau)|\,\mathrm{e}^{\nu\sigma}\sin(\nu t + \tfrac{1}{4}\pi), \quad |\arg \nu| < \pi \qquad (7b)$$

$$\sim |V(\tau)|\,\mathrm{e}^{\nu\sigma}\{\sin(\nu t + \tfrac{1}{4}\pi) - \cot \nu\pi \cos(\nu t + \tfrac{1}{4}\pi)\}, \quad |\arg(-\nu)| < \pi \qquad (7c)$$

where t is defined by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{2\tau}(M^2-1)^{1/2}, \quad t = 0 \quad \text{for} \quad \tau = \frac{\gamma-1}{\gamma+1}$$

2.2 *The hodograph transformation, incompressible flow*

Thus far, we have reviewed the more advantageous features of the hodograph theory: a non-linear differential equation has been transformed into a linear one, possessing a convenient set of particular solutions having easily accessible analytical properties. However, there are also serious drawbacks, mainly stemming from the complication under the hodograph transformation of the topological structure of the solution being described.

As an example, consider the flow which will be the starting point for our further developments: the incompressible circulatory potential flow around an elliptical cylinder.

This is given by:

$$\Phi(z) = z_1 + \frac{1}{z_1} + i\Gamma \log z_1 \tag{8a}$$

$$z = z_1 + \frac{\varepsilon}{z_1} \tag{8b}$$

Now introducing the conjugate velocity

$$\zeta = \frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{\mathrm{d}\Phi}{\mathrm{d}z_1} \Big/ \frac{\mathrm{d}z}{\mathrm{d}z_1} \tag{9}$$

we have one of the few cases in which the hodograph transformation can be obtained simply by algebraic elimination of z_1 from eqs. (8) and (9), and the complex potential in the hodograph variable ζ can be written:

$$\begin{aligned}\Phi = &\left(\frac{1}{1-\zeta} + \frac{1}{1-\varepsilon\zeta}\right)\varepsilon^{1/2}\{(\zeta_1-\zeta)(\zeta_2-\zeta)\}^{1/2} \\ &+ \int_\infty^\zeta \left(\frac{1}{1-\zeta} + \frac{\varepsilon}{1-\varepsilon\zeta}\right)\varepsilon^{-1/2}\frac{\Gamma^2}{4}\{(\zeta_1-\zeta)(\zeta_2-\zeta)\}^{-1/2}\,\mathrm{d}\zeta \\ &+ \frac{i\Gamma}{2}\left\{-\frac{1}{1-\zeta} - \log(1-\zeta) + \frac{1}{1-\varepsilon\zeta} + \log(1-\varepsilon\zeta)\right\}\end{aligned} \tag{10}$$

with $$\zeta_1, \zeta_2 = \frac{1+\varepsilon}{2\varepsilon} \mp \frac{1+\varepsilon}{2\varepsilon}\left\{1 - \frac{4\varepsilon}{(1+\varepsilon)^2}\left(1 - \frac{\Gamma^2}{4}\right)\right\}^{1/2}$$

Analysis of this function shows that the basic structure of the solution is given by a Riemann surface of two sheets, with ζ_1, ζ_2 as first-order branch points. (For $0 < \Gamma < 2$, $|\varepsilon| < 1$ we have $|\zeta_1| < 1$, $|\zeta_2| > |1/\varepsilon|$). In one of the sheets, the potential Φ has a singularity of the type

$$-i\Gamma\left\{\frac{1}{1-\zeta} + \log(1-\zeta)\right\}$$

(dipole + vortex) at $\zeta = 1$, and Φ is regular at $\zeta = 1/\varepsilon$. In the other sheet, the potential is regular at $\zeta = 1$, and has a singularity of strength

$$i\Gamma\left\{\frac{1}{1-\varepsilon\zeta} + \log(1-\varepsilon\zeta)\right\}$$

at the point $\zeta = 1/\varepsilon$. It follows that $\zeta = 1$ and $\zeta = 1/\varepsilon$ are branch points of logarithmic type for the complex potential Φ; for the stream function, however, these points are not branch points as the periods $(\pm 2\pi\Gamma)$ are real.

Note that as $\Gamma \to 0$, the singularities at $\zeta = 1$, $\zeta = 1/\varepsilon$ become first-order branch points by confluence with ζ_1, ζ_2.

A sketch of the mapping between the physical plane and the hodograph has been prepared in Fig. 1 for the case of fore and aft symmetery (ε real). The branch point ζ_1 in the hodograph corresponds to a saddle point for the isochores and isogonals in the physical plane, and the point $\zeta = 1$, of course, corresponds to infinity in the physical plane. The singularities $\zeta = 1/\varepsilon$ and ζ_2

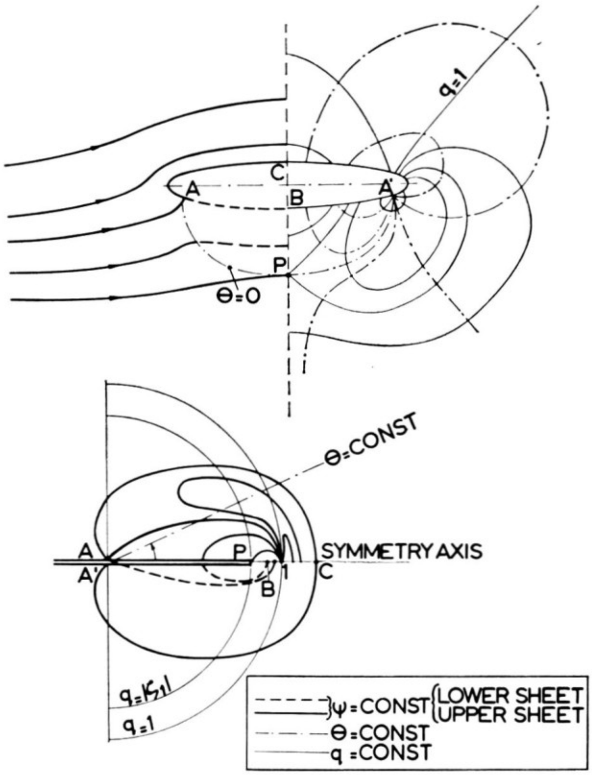


FIG. 1 — Hodograph of symmetrical circulatory incompressible flow around elliptical cylinder; streamlines, isochores and isogonals

belong to the interior flow field of the ellipse, which has not been drawn. Note that the part *PABA'P* of the physical plane is mapped on the 'second' sheet of the hodograph manifold.

2.3. *Construction of an integral operator*

We now return to consider Chaplygin's original idea: the construction of a compressible flow related to an incompressible one, by exploiting the similarity between what are in fact particular solutions of Laplace's equation: $\zeta^{\nu}=q^{\nu}e^{-i\nu\theta}$ and the Chaplygin particular solutions $\psi_{\nu}e^{-i\nu\theta}$. That is we want to construct an operator, mapping the analytic potential of a given incompressible flow around a body into the vector space of the Chaplygin particular solutions of the hodograph equation. This operator is to be continuously dependent on the parameter τ_1, in such a way that for $\tau_1 \to 0$ the given flow is recovered. Now probably the most logical basis for the construction of this operator is a Mellin transform theorem for analytic functions (*see* ref. 14, th. 31).

Roughly, let $\Phi(\zeta)$, $\zeta=qe^{-i\theta}$, be an analytic function, regular in the sector

$$-\alpha < \arg(-\zeta) < \beta, \quad 0 \leqq (\alpha, \beta) < \pi$$

and let

$$\Phi(\zeta) = O(|\zeta|^{a}) \quad \text{for} \quad |\zeta| \to \infty$$

$$\Phi(\zeta) = O(|\zeta|^{b}) \quad \text{for} \quad |\zeta| \to 0, \quad a < b$$

in this sector (we write $\Phi \in M(\alpha, \beta; a, b)$).

Then the Mellin transform

$$\mathcal{M}\{\Phi\} \equiv \int_{-\infty}^{0} \Phi(-\zeta)^{-\nu-1}\,d\zeta \tag{11}$$

is holomorphic in the strip $a < \text{Re}\,\nu < b$ of the ν-plane, and

$$\Phi = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} (-\zeta)^{\nu} d\nu \int_{-\infty}^{0} \Phi(-\zeta)^{-\nu-1}\,d\zeta, \quad a < k < b \tag{12}$$

Now consider expressions of the form:

$$\tilde{\psi} = \text{Im}\,\tilde{\Phi}(\tau, \theta; \tau_1) \tag{13}$$

$$\tilde{\Phi} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \psi_{\nu}(\tau) f_{\nu}(\tau_1)\, e^{i\nu(\mp\pi-\theta)}\, d\nu \int_{-\infty}^{0} \Phi(-\zeta)^{-\nu-1}\,d\zeta,$$

$$\Phi \in M(\alpha, \beta; a, b), \quad a < k < b, \quad |\mp\pi-\theta| < \pi$$

which are obviously solutions of eq. (1) in domains of uniform convergence of the integral.

To ensure convergence, the integrand in eq. (13) must be provided with an asymptotic behaviour as a function of ν equivalent to that in eq. (12); furthermore we require $f_\nu(\tau_1)$ to be $\sim\tau_1^{-\nu/2}$ for $\tau_1\to 0$, so that $\tilde{\Phi}\to\Phi$, and $\tau_1<(\gamma-1)/(\gamma+1)$.

As we have

$$|\zeta|^\nu=\exp[\nu(\log q-\log q_\infty)],\quad q_\infty=1$$

and

$$\psi_\nu(\tau)\sim V(\tau)\,e^{\nu s}\qquad \text{for}\qquad \tau<\frac{\gamma-1}{\gamma+1}$$

we want $f_\nu(\tau_1)$ to be a meromorphic function of ν with simple poles only ($\nu=0,\,-1,\,-2,\ldots$ being excluded), having the asymptotic behaviour for large ν

$$f_\nu(\tau_1)\sim A(\tau_1)\,e^{-\nu s_1} \tag{14}$$

Then, the integrands in eqs. (12) and (13) have similar asymptotic behaviour for large ν, and we have $\tilde{\Phi}\to\Phi$ for $\tau_1\to 0$ by eq. (4).

We have now obtained (except for the further determination of $f_\nu(\tau_1)$) a solution of eq. (1), related in the way required to an analytic function, and defined in the sector of regularity of the original function for

$$0<\tau<\frac{\gamma-1}{\gamma+1}.$$

Starting from eq. (13), we can construct the expression for the co-ordinates in the physical plane corresponding to the stream function Im $\tilde{\Phi}(\tau,\theta;\tau_1)$, by using the definition of stream function and velocity potential in the compressible flow, and the Chaplygin equations. The result is:

$$\tilde{z}=\left(\frac{\tau_1}{\tau}\right)^{1/2}\frac{\tau}{(1-\tau)^{1/(\gamma-1)}}$$
$$\times\left[\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{e^{\mp\pi i\nu}}{\nu-1}e^{-i(\nu-1)\theta}\left(\psi_\nu'(\tau)+\frac{\nu}{2\tau}\psi_\nu(\tau)\right)f_\nu(\tau_1)\,d\nu\int_{-\infty}^{0}\Phi(-\zeta)^{-\nu-1}\,d\zeta\right.$$
$$\left.-\overline{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{e^{\mp\pi i\nu}}{\nu+1}e^{-i(\nu+1)\theta}\left(\psi_\nu'(\tau)-\frac{\nu}{2\tau}\psi_\nu(\tau)\right)f_\nu(\tau_1)\,d\nu\int_{-\infty}^{0}\Phi(-\zeta)^{-\nu-1}\,d\zeta}\right]$$
$$\Phi\in M(\alpha,\beta;a,b),\quad a<k<b \tag{15}$$

where the prime denotes differentiation with respect to τ, and the bar indicates the complex conjugate.

2.4. *Analytical continuation; determination of $f_\nu(\tau_1)$*

The next problem is the analytic continuation of the solution out of the original sector of regularity around the singularities that determine the hodograph of the flow around a body, an example of which has already been discussed. Here we can proceed in two different directions.

In the first place it is possible to localize the transformation, thus removing the restriction $\Phi \in M$ altogether, and make any arbitrary analytical function admissible as an original. This leads to the construction of an integral operator in the sense of Bergman; the derivation of the general operators constructed by Lighthill[8] from our eqs. (13), (15), is quite straightforward[13].

Now, from the analytical point of view these operators, which can be defined in the physical plane of an arbitrary incompressible potential flow, are the most general possible. However, they are restricted from the physical point of view in admitting subsonic flow speeds only. Lighthill indicated a somewhat devious way to obtain transonic flow solutions from this operator for analytic hodograph potentials of a certain type. This method was followed in an earlier paper[15].

For the much more complicated work on the circulatory flow case, a more direct way is preferable. This is, in fact, a mathematical interpretation and generalization of the method employed by Goldstein, Lighthill and Craggs[4], and Lighthill[6], in their analysis of the compressible flow related to the incompressible one around the circular cylinder. This method can be described as follows.

First, when $\Phi(\zeta)$ is a hypergeometric function with singular points $(0, 1, \infty)$, we know that a Mellin transform eq. (11) exists as a meromorphic function defined in the entire ν-plane except for a countable set of simple poles, and the inversion integral eq. (12) is well known as the Barnes integral representation for the hypergeometric function. This integral representation can be used to construct the continuation of the hypergeometric series convergent for $|\zeta|<1$ into series convergent for $|\zeta|>1$, and in fact to continue the initial series on the whole Riemann surface underlying the hypergeometric function. However, as by construction, we have similar asymptotic properties in the ν-plane for the integrands of eqs. (12) and (13), we can equally well evaluate the integral eq. (13), provided the contributions of the additional poles of $\psi_\nu(\tau)f_\nu(\tau_1)$ are taken into account.

It is clear that we can now use the whole machinery of the Mellin transform theory to generalize this procedure, in particular we can consider functions, constructed by a finite number of operations (sum, product, differentiation, etc.) from a finite number of generalized hypergeometric functions, each singular at a different triple of points $(0, \zeta_p, \infty)$.

For such functions Φ, which constitute a sub-set $M_H \subset M$, the Mellin transforms can be built up from the elementary transforms by the corresponding operations in the transform theory.

$\Phi(\zeta)$ is then a function singular at a finite number of isolated points in the ζ-plane, and the continuations around these singularities can be constructed in a similar way as for the hypergeometric function.

The resulting integrals of the type eq. (13), being evaluated by the residue theorem, yield series involving the Chaplygin functions, the convergence of which can be investigated using the known asymptotic properties of these functions (eqs. 7). The whole point of this alternative procedure is now, that while the operator eq. (13) is defined for subsonic values of τ only, part of the resulting series solution can be continued trivially into the supersonic domain.

Formally, these two methods of analytic continuation (one leading to Lighthill's integral operator, the other using explicit Mellin transforms) compare with each other as a reversal in the order of integration.

Two methods have been indicated in this paragraph to continue analytically the originally given solution eq. (13) out of its sector of regularity. Both methods have been used by Lighthill[6, 8] to determine the function $f_\nu(\tau_1)$. The condition is that the stream function returns to its original value, when going round the body in the physical plane of the compressible flow, or when describing the corresponding contour in the hodograph.

From this analysis it follows that in the non-circulatory flow case, a class of functions $f_\nu(\tau_1)$ is admissible (admitting of the conditions given with eq. (14)), of which

$$f_\nu(\tau_1) \equiv e^{-\nu s_1} \tag{16}$$

is the simplest.

In the circulatory flow case, however, a unique choice is dictated by the properties of the dipole/vortex singularity in the hodograph as described in section 2.2, and we have:

$$f_\nu(\tau_1) \equiv \frac{\psi_{-\nu}(\tau_1) + 2\tau_1 \psi'_{-\nu}(\tau_1)}{1-\nu} \tag{17}$$

3. Compressible Flow around Quasi-elliptical Aerofoil Sections

3.1. *Introduction*

The foregoing theory is now to be applied to obtain compressible flows related to the incompressible flow around an elliptic cylinder. As we are mainly interested in transonic flows, the second method described, using explicit Mellin transforms, is to be used. This results in a theory, simple in principle, but especially in the circulatory flow case already rather more complicated from the computational point of view than one would like. Still, this represents only one particular family of compressible flows, and one would be inclined to be immediately interested in further developments, based on more general incompressible flows. Unfortunately, this possibility is severely restricted by the necessity of having explicit expressions for the complex potential in the hodograph available, which moreover must have a very special analytical structure.

However, there is a second, much more simple way of generalizing the solutions obtained: the theory being a linear one, we can add further particular solutions, and so we have an infinite number of additional parameters at our disposal. In fact, the family of quasi-elliptical aerofoil sections is astonishingly varied, and we have only just begun to explore its possibilities in the non-circulatory flow case.

3.2. *Quasi-elliptical aerofoil flows: integral representations*

We will now present explicit expressions for the Mellin transform of the hodograph potential of the flow around an elliptical cylinder. These expressions, substituted into the general inversion integrals for the stream function and the co-ordinates (eqs. 12 and 15), and using eq. (17) for $f^{\nu}(\tau_1)$ (or eq. (16) in the zero lift case), represent the compressible flow solution in a sector of the hodograph for subsonic flow speeds. How to obtain from this series expansions representing the complete solution and permitting extension into the transonic domain, will in section 3.3 only be indicated for the simplest (non-circulatory) case.

Putting $\Gamma=0$ in eq. 10 we have

$$\Phi_0(\zeta) = \frac{2-(1+\varepsilon)\zeta}{\{(1-\zeta)(1-\varepsilon\zeta)\}^{1/2}} \tag{18}$$

and it is sufficient to consider

$$I_0 = \{(1-\zeta)(1-\varepsilon\zeta)\}^{-1/2} \tag{18a}$$

This is clearly a product of two hypergeometric functions, and we have

$$\begin{aligned} \mathscr{M}\{I_0\} &\equiv \int_{-\infty}^{0} I_0(-\zeta)^{-\nu-1}\,\mathrm{d}\zeta \\ &= \frac{(-\nu-1)!\,(\nu-\frac{1}{2})!}{(-\frac{1}{2})!} F(\tfrac{1}{2}, -\nu; \tfrac{1}{2}-\nu; \varepsilon) \\ &+ \varepsilon^{\nu+1/2}\frac{\nu!(-\nu-\frac{3}{2})!}{(-\frac{1}{2})!} F(\tfrac{1}{2}, \nu+1; \nu+\tfrac{3}{2}; \varepsilon) \end{aligned} \tag{19}$$

Note that this expression has poles at the integers only: the poles at

$$\nu = \pm N + \tfrac{1}{2} \quad (N \text{ integer})$$

in the separate terms just cancel each other out. Accordingly, the Mellin transform is regular in the strip $-1 < \operatorname{Re}\nu < 0$, as it should be.

The Mellin transform for the full eq. (10) becomes:

$$\begin{cases} \mathcal{M}\{\Phi\} \equiv (\zeta_1\zeta_2)^{1/2}\varepsilon^{1/2}\Bigg\{\sum_{m=0}^{\infty}(-1)^m(1+\varepsilon^m)\dfrac{(-\nu+m-1)!}{(-\frac{3}{2})!}(\nu-m-\tfrac{3}{2})! \\ \qquad\times F\left(-\tfrac{1}{2}, -\nu+m; -\nu+m+\tfrac{3}{2}; \dfrac{\zeta_1}{\zeta_2}\right)\zeta_1^{m-\nu} \\ +\sum_{m=0}^{\infty}(-1)^m(1+\varepsilon^{-m-1})\dfrac{(\nu+m-1)!}{(-\frac{3}{2})!}(-\nu-m-\tfrac{3}{2})! \\ \qquad\times F\left(-\tfrac{1}{2}, \nu+m; \nu+m+\tfrac{3}{2}; \dfrac{\zeta_1}{\zeta_2}\right)\left(\dfrac{\zeta_1}{\zeta_2}\right)^{-1/2}\zeta^{-\nu-m-1}\Bigg\} \\ -\dfrac{\Gamma}{2}(1+\varepsilon^{\nu})\dfrac{\pi}{\cos\pi\nu} \end{cases}$$

$$\begin{cases} -\dfrac{\Gamma^2}{4}(\zeta_1\zeta_2)^{-1/2}\varepsilon^{-1/2}\Bigg\{\dfrac{1}{\nu}\sum_{m=0}^{\infty}(-1)^m(1+\varepsilon^{m+1})\dfrac{(-\nu+m)!}{(-\frac{1}{2})!}(\nu-m-\tfrac{3}{2})! \\ \qquad\times F\left(\tfrac{1}{2}, -\nu+m+1; -\nu+m+\tfrac{3}{2}; \dfrac{\zeta_1}{\zeta_2}\right)\zeta_1^{-\nu+m+1} \\ +\dfrac{1}{\nu}\sum_{m=0}^{\infty}(-1)^m(1+\varepsilon^{-m})\dfrac{(\nu+m)!}{(-\frac{1}{2})!}(-\nu-m-\tfrac{3}{2})! \\ \qquad\times F\left(\tfrac{1}{2}, \nu+m+1; \nu+m+\tfrac{3}{2}; \dfrac{\zeta_1}{\zeta_2}\right)\left(\dfrac{\zeta_1}{\zeta_2}\right)^{1/2}\zeta^{-m-\nu}\Bigg\} \\ +\dfrac{\Gamma}{2}(1+\varepsilon^{\nu})\dfrac{1}{\nu}\dfrac{\pi}{\cos\pi\nu} \end{cases}$$

$$\left\{ +\frac{i\Gamma}{2}\left(1-\frac{1}{\nu}\right)(1-\varepsilon^{\nu})\frac{\pi}{\sin\pi\nu} \right. \tag{20}$$

Again in this expression, poles occur only at the integers, the contributions of the broken values of ν cancelling out.

3.3. *Series expansions and analytical continuation in the zero lift case*; $f_\nu(\tau_1) = e^{-\nu s_1}$

Now consider the simplest case in the theory of quasi-elliptical aerofoils, which can be represented by:

$$\psi = \operatorname{Im}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi_\nu(\tau)\, e^{\nu[-s_1+i(\mp\pi-\theta)]}\,d\nu\,\mathcal{M}\{I_0\} \tag{21}$$

$\mathcal{M}(I_0)$ being given by eq. (18*a*).

(An equivalent solution was given by LEVEY[(16)], using Cherry's theory, and by the author in ref. 13, starting from the operator form of Lighthill's theory.)

The integral can be evaluated by summing the residues of the poles in the right-hand ν-half plane, and one obtains the series

$$L_1 \equiv \sum_{n=0}^{\infty} \lambda_{1,n}\psi_n(\tau)\,e^{-n(s_1+i\theta)} \tag{22}$$

where
$$\lambda_{1,n} = \frac{(n-\frac{1}{2})!}{(-\frac{1}{2})!\,n!}F(\tfrac{1}{2},-n;\tfrac{1}{2}-n;\varepsilon)$$

and we have to take the imaginary part to obtain the stream function. This series can be shown to converge for $\tau<\tau_1$. The analytic continuation of this series is now obtained by summing the residues of the poles in the left-hand ν-half plane for the part of the integrand corresponding to the first term in $\mathcal{M}\{I_0\}$, and by summing in the right-hand ν-half plane for the second term.

The resulting series can be written for $-\theta \gtrless 0$ respectively

$$\pm L_2 + L_2^* \equiv \pm\sum_{n=0}^{\infty}\Big\{\lambda_{2,n+1/2}\psi_{n+1/2}(\tau)\,e^{-(n+1/2)(s_1+i\theta)} + \lambda_{2,-n-1/2}\psi_{-n-1/2}(\tau)\,e^{(n+1/2)(s_1+i\theta)}\Big\} + \sum_{n=2}^{\infty}\lambda^*_{2,-n}\,.\,-nC_n\psi_n(\tau)\,.\,e^{h(s_1+i\theta)} \tag{23}$$

where
$$\lambda_{2,n-1/2} = i\frac{(n-\frac{1}{2})!}{(-\frac{1}{2})!\,n!}F(\tfrac{1}{2},n+\tfrac{1}{2};n+1;\varepsilon)$$

$$\lambda_{2,n+1/2} = i\varepsilon^{n+1/2}\frac{(n+\frac{1}{2})!}{(-\frac{1}{2})!(n+1)!}F(\tfrac{1}{2},n+\tfrac{3}{2};n+2;\varepsilon)$$

$$\lambda^*_{2,-n} = \frac{\pi(n-1)!}{(-\frac{1}{2})!(n-\frac{1}{2})!}F(\tfrac{1}{2},n;n+\tfrac{1}{2};\varepsilon)$$

The first of this series is obtained by summing over the residues of the poles of the Mellin transform, and converges for $\tau>\tau_1$, at least when τ_1 is sufficiently large (more precisely, when $e^{\sigma-s_1}<|1/\varepsilon|$). The second series is the contribution of the additional poles of $\psi_\nu(\tau)$ in the left-hand ν-half plane, and converges for all $0\leqq\tau<1$. When we let $q_{max}\rightarrow\infty$, the L_2 series reduces to the corresponding series in the incompressible flow case, and the L_2^* series vanishes

For sufficiently large values of q_{max}, we can also study the continuation into the region corresponding to the region $|\zeta|>|1/\varepsilon|$ in the incompressible flow, by closing the contour of integration in the left-hand ν-half plane. In all cases of practical interest, however, this region merely describes the continuation of the flow into the interior of the aerofoil, and will therefore here be ignored.

At this point, we still have to obtain the series expansions for the complete aerofoil flow by writing for the coefficients (*cf.* eqs. (18), (18*a*)):

$$c_{i,p} = 2\lambda_{i,p} - (1+\varepsilon)\lambda_{i,p-1}, \quad c_{1,0} = 2 \tag{24}$$

We can now easily work out the contribution of the initial series L_1 on the two-sheeted Riemann surface underlying the compressible flow solution, the result is represented in Fig. 2.

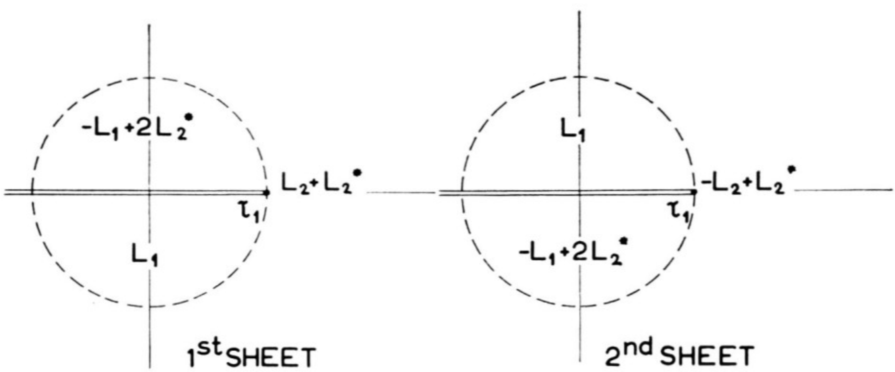


FIG. 2 — Analytical continuation of series solution Eq. (22), (23) on two-sheeted Riemann surface

3.4. *Examples*

We will now present some explicit examples of the flows obtained, simultaneously demonstrating some of the possibilities of generalizing the solutions.

The series L_2^* in eq. (23) being convergent on the whole hodograph, can be added to the initial series and its analytic continuations, and we consider a three parameter (τ_1, ε, λ) family of symmetrical aerofoils by choosing ε real and the series $L_1+(\lambda-1)L_2^*$ as the commencing branch of the solution. For $\lambda=1$, the solution discussed in the previous paragraph is recovered, for $\lambda=0$ one obtains a symmetrical aerofoil, having in addition fore and aft symmetry. It appears (*see* ref. 15 for a further discussion) that for τ sufficiently high, this doubly symmetrical aerofoil will have cusped ends (Fig. 3). Choosing $\lambda \neq 0$ one obtains aerofoil sections having a stagnation point at the leading edge and a cusped trailing edge, in fact the leading edge nose radius can be controlled by λ (Fig. 4). For certain choices of λ, one can obtain pressure distributions, which have the 'peaky' character shown by Pearcey to

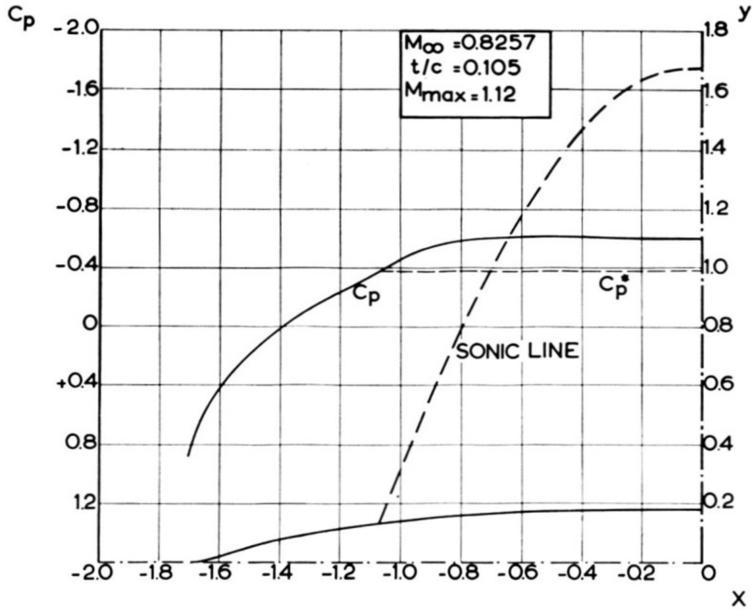


FIG. 3 — Doubly symmetrical quasi-elliptical aerofoil section, $\tau_1=0{\cdot}12$, $\epsilon=0{\cdot}7$, $\lambda=0$ (Eqs. (22), (23))

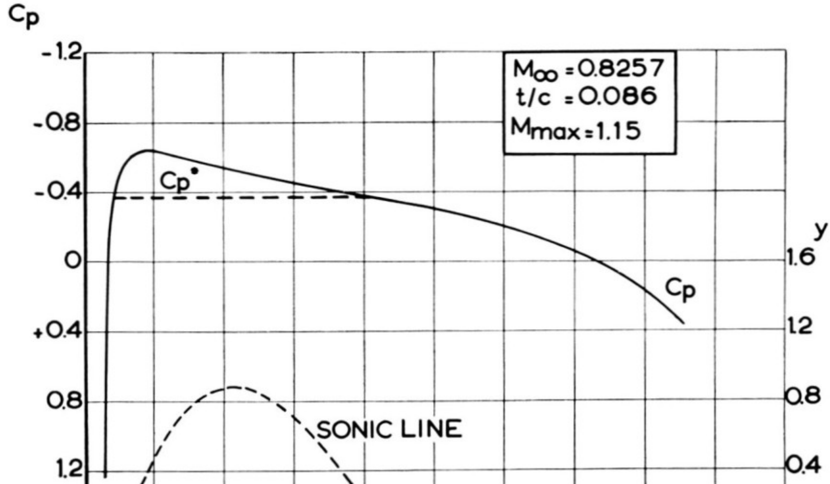


FIG. 4 — Symmetrical quasi-elliptical aerofoil section, $\tau_1=0{\cdot}12$, $\epsilon=0{,}8$, $\lambda=1{\cdot}15$ (Eqs. (22), (23))

be conducive to a shock-free real flow; a good example is given in Fig. 5, where we have sketched in the low speed pressure distribution for comparison.

Further increasing λ for fixed τ_1 and ε, the interpretability of the solution is finally destroyed by a limit line singularity piercing through the aerofoil contour.

It is also possible to produce cambered (but non-lifting) aerofoils. In Fig. 6, an example is given of the linearised effect of adding the stream function $+0{\cdot}16\psi_2 \cos 2\theta$ to the $\tau_1=0{\cdot}11$, $\varepsilon=0{\cdot}75$, $\lambda=0{\cdot}9$ aerofoil. The non-linear effect of this addition destroys (on a micro-scale) the closure of the aerofoil both at the leading and trailing edge, this must be corrected for by adding a further compensating solution (*see* ref. 13).

For the circulatory flow case, we very recently obtained our first result: the slightly compressible flow ($M_\infty=0{\cdot}2$) around an about 50% thick aerofoil. Here, also, there is some closure difficulty at the trailing edge requiring corrections. For the present a more practical example is a non-lifting aerofoil computed using the integral transformation to be used in the lifting case, i.e. using the $f_\nu(\tau_1)$ given in eq. 17. As is shown in Fig. 7, this is quite a healthy-looking specimen, which leads one to expect that the shapes obtained in the circulatory flow case will also be satisfactory from the practical point of view.

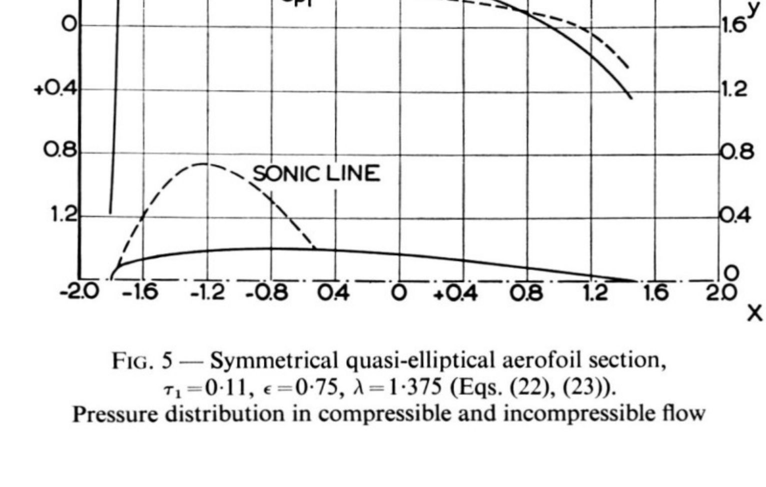


FIG. 5 — Symmetrical quasi-elliptical aerofoil section, $\tau_1=0{\cdot}11$, $\epsilon=0{\cdot}75$, $\lambda=1{\cdot}375$ (Eqs. (22), (23)). Pressure distribution in compressible and incompressible flow

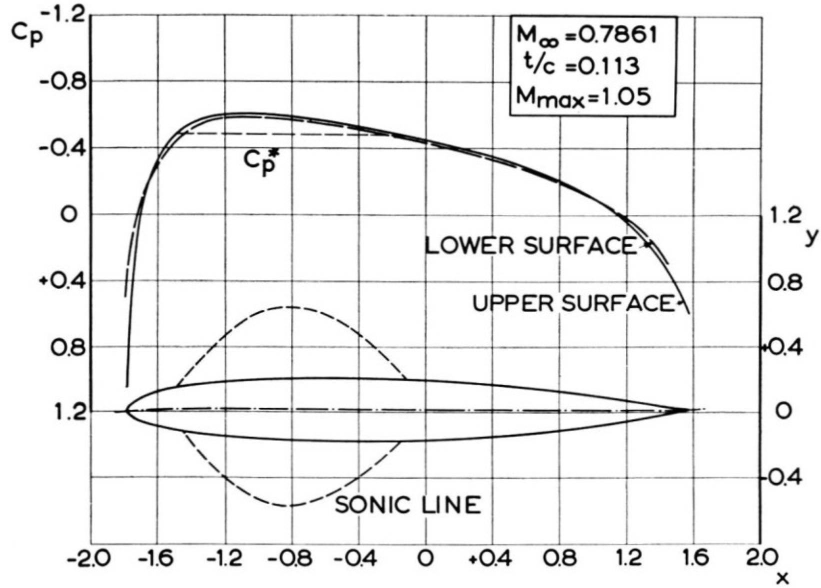


FIG. 6 — Asymmetrical quasi-elliptical aerofoil section, $\tau_1 = 0{\cdot}11$, $\epsilon = 0{\cdot}75$, $\lambda = 0{\cdot}9$ (Eqs. (22), (23) + $\Delta\psi = 0{\cdot}16\ \psi_2(\tau) \cos 2\theta$)

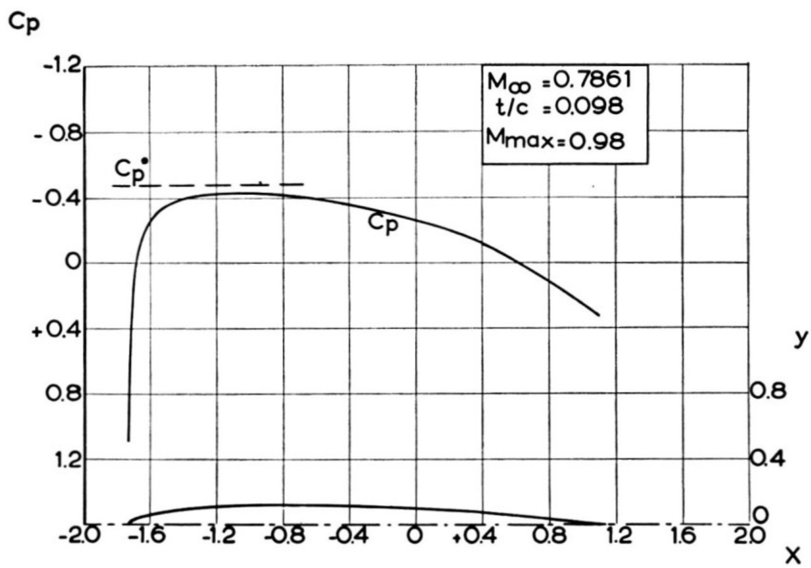


FIG. 7 — Symmetrical quasi-elliptical aerofoil section, $\tau_1 = 0{\cdot}11$, $\epsilon = 0{\cdot}75$ (Eqs. (13), (17), (18))

4. Physical Aspects: the Transonic Controversy

Up to this point, the subject-matter announced in the title of this paper: the design of shock-free transonic flows, has not been made good, as we have merely been discussing methods of computing transonic potential flows. Now, what guarantee is there of the aerofoil shapes theoretically obtained producing a shock-free real flow?

This question is currently being investigated in an experimental programme based on a family of quasi-elliptical aerofoils, which is performed in collaboration with the National Physical Laboratory. Until this research has been completed, the answer to this question is, that we do not really know. However, the fact that some of the flows obtained exhibit the characteristics of the aerofoils proved to be virtually shock-free in Pearcey's transonic experiments, leads one to expect that these might be satisfactory from this point of view; on the other hand there is good reason to suspect that other shapes, defined by the same family of potential flow solutions, will in fact generate a shocked flow. What is clearly required is some criterion, based on a physical understanding of the flow, which would separate the 'sheep from the goat'. This brings us right back into the middle of the old 'transonic controversy' already touched upon in the introduction.

Now what made this discussion at times somewhat confusing is that there are three points of view involved: the mathematician's, the physicist's and the engineer's, and that these are all relevant but should not be mixed up.

The most precise statements came, of course, from the mathematical side: the famous non-existence theorems of Mrs. Morawetz. One of these(17) states, roughly, that a transonic potential flow solution has no neighbouring potential flow solutions for an arbitrary, small change of the boundary in the supersonic region. The mathematical implication of this already classical result is clear: the boundary value problem for the equations of transonic potential flow around a given shape has in general no solution. (Note that the solutions discussed in this paper are defined in a quite different way, on the other hand these define only one particular family of shapes.)

However, it is difficult to see how to come to any definite physical conclusion from these mathematical facts. The unqualified assertion that a transonic potential flow has no physical significance would seem to be bordering on the trivial: strictly speaking potential flows as such never have physical significance without interpretation in the context of a physically more complete theory (involving viscosity and unsteadiness); or to be downright false if it is meant to imply that transonic flows are necessarily dominated by shock waves. In other words: to draw a strict physical conclusion from Morawetz's results would seem to be impossible, just because a potential flow model does not contain enough of the relevant physics of the flow.

The aeronautical engineer is, of course, quite used to working in conditions in which the mathematical and physical aspects are not completely understood, we need only mention turbulence, or separation. In fact, perhaps the essence of his art consists in controlling such only partially mapped areas of physical knowledge. However, he is not very likely to be interested in discussions on such things as 'a shock wave which is . . . so weak as to be unobserved'. From his point of view, shock waves are important in terms of shock-induced boundary layer separation, transonic drag rise, etc. It is in this practical sense that the term 'shock-free transonic flow' is to be understood, and such flows have been experimentally realised by Pearcey. For the engineer transonic potential flow solutions may be useful in the way the mathematical notion of potential flow is always used in practical aerodynamics: not as having predictive physical significance, but as representing an ideal (loss free) reference to be approximated by careful design.

It remains, however, to elucidate from the physical point of view under what conditions, in a practical sense, such shock-free flows can occur. At the N.L.R., we are investigating this stability problem from the point of view that the key may be in the time history of moving (weak) shock waves originating in the flow, and we have designed a number of experiments to study the interaction of 'shock-free' transonic flows and acoustic fields(18).

It would appear, then, that we have now both the theoretical and experimental tools available to approach these longstanding problems from a systematic basis. The current interest in the aerodynamic optimization of swept winged long range transports, both in the high subsonic and supersonic speed range, would seem to make this a not merely academic exercise.

References

(1) Molenbroek, P., 'Ueber einige Bewegungen eines Gases bei Annahme eines Geschwindigkeits-potentials.' *Arch. Math. Phys.* **9**, pp. 157–195 (1890).

(2) Chaplygin, S. A., 'On gas jets.' *Sci. Mem. Moscow. Un. Math. Phys.* no. 21, pp. 1–21 (1904). Transl. NACA TM 1063 (1944).

(3) Karman, T. von, 'Compressibility effects in aerodynamics.' *J.A.S.*, **8**, pp. 337–356 (1941).

(4) Goldstein, S., Lighthill, J. M. and Craggs, J. W., 'On the hodograph transformation for high speed flow I.' *Q. J. Mech. Appl. Math.*, **1**, pp. 344–357 (1948).

(5) Cherry, T. M., 'Flow of compressible fluid about a cylinder.' *Proc. R. Soc. London, A* 192, pp. 45–79 (1947).

(6) Lighthill, M. J., 'On the hodograph transformation for high speed flow II. A flow with circulation.' *Q. J. Mech. Appl. Math.*, **1**, pp. 442–450 (1948).

(7) Cherry, T. M., 'Flow of a compressible fluid about a cylinder II. Flow with circulation.' *Proc. R. Soc. London, A*, 196, pp. 1–31 (1949)

(8) Lighthill, M. J., 'The hodograph transformation in transonic flow. (II) Auxiliary theorems on the hypergeometric functions. (III) Flow round a body.' *Proc. R. Soc. London, A* 191, pp. 341–369 (1947).

(9) KRZYWOBLOCKI, M. Z., *Bergman's linear integral operator method in the theory of compressible fluid flow*. Berlin (1960).
(10) BUSEMANN, A., 'The drag problem at high subsonic speeds.' *J.A.S.*, **16,** pp. 337–344 (1949).
(11) HOLDER, D. W., 'Transonic flow past two dimensional aerofoils.' *J.R.Aer.Soc.*, **68,** pp. 501–516 (1964).
(12) PEARCEY, H. H., 'The aerodynamic design of section shapes for swept wings.' In: *Advances in Aeronautical Sciences*, **3.** London (1962).
(13) NIEUWLAND, G. Y., 'Transonic potential flow around a family of quasi-elliptical aerofoil sections.' N.L.R. report (*to be published*).
(14) TITCHMARSH, E. C., *Introduction to the theory of Fourier integrals.* Oxford (1937).
(15) NIEUWLAND, G. Y., 'The computation by Lighthill's method of transonic potential flow around a family of quasi-elliptical aerofoils.' NLR-TR T. 83 (1964).
(16) LEVEY, H. C., 'High speed flow of a gas past an approximately elliptic cylinder.' *Proc. Cambr. Phil. Soc.*, **46** (1950).
(17) MORAWETZ, C. S., 'On the non-existence of continuous transonic flows past profiles II.' *Comm. Pure Appl. Math.*, **10,** pp. 107–131 (1957).
(18) SPEE, B. M., 'Wind tunnel experiments on unsteady cavity flow at high subsonic speeds.' Agard CP, no. 4 (1966).

DISCUSSION

H. H. Pearcey (National Physical Laboratory, Teddington, Middx., England): Mr. Nieuwland indicated that at the N.P.L. we had succeeded in achieving transonic flows that were essentially shock-free and that I had promised to present some of this experimental evidence on this occasion. Before doing this, however, I would like to take the opportunity of congratulating Mr. Nieuwland on a very fine piece of work. I have had the privilege of being associated with him over the past two or three years during the progress of his research, and in my opinion this work is the most significant application yet of mathematical theory to the solution of problems of transonic flows for aerofoils that are of practical significance in aircraft design. I mean here, of course, aerofoils that have a finite thickness somewhere between the flat plate and circular cylinder that have hitherto been so popular with mathematicians, aerofoils that have round leading edges, and aerofoils that carry lift. In concentrating his lecture on one of the most interesting and important aspects of his research, namely, the derivation of aerofoils that produce shock-free compressions from local supersonic flow, he modestly ommitted to describe other aspects of his work that will, I am sure, be of great practical significance too. He has provided us with exact solutions for sub-critical flows on symmetrical aerofoils and will shortly do the same for lifting aerofoils. These exact solutions will form the basis for consolidating the more approximate methods on which aircraft engineers will be able to rely for many years to come in wing-section design.

The work on these problems of transonic flows has, for many years, been going in 'fits and starts' or, in more with-it phraseology, has been characterised by 'stop and go'. I think we now have the signal for 'all systems go', and I hope that this will lead to the start of a steady and productive stream of research and design.

There are too many reasons to discuss in a brief contribution why the situation has been a stop and go one for so long. One of these reasons has undoubtedly been the extreme mathematical difficulties that have prevented the most eminent mathematicians from deriving solutions that even approach the physical realities of the situation. We have had to wait for the rare combination of abilities that Mr. Nieuwland has brought to bear on this problem in such a timely fashion. He has handled brilliantly the mathematical complexities in extending the framework provided by his eminent predecessors. He has exploited to the full the possibilities of numerical methods that can now be used on modern digital computers. He has shown a remarkable persistence in pressing this work to a fruitful outcome. Above all, he has directed his efforts in a physically realistic way, and in this connection I would like to pay tribute to the insight that he has given me on the physical nature of some of these problems.

In the absence of the mathematical tools that he is now providing us, some of us have had to stumble along as best we could experimentally and empirically to solve practical problems of aircraft design. I now hope that we can proceed in collaboration with him in a less painful and more rational fashion.

To encourage him in pressing on with his research, and others to join us, I would now like to produce some of the many experimental results that show that effectively shock-free flows are possible and that, more generally, isentropic compressions can be exploited to reduce the strength of shock waves when they do occur.

The first example that I have selected (Fig. 1) is one in which we can compare a flow that has an essentially shock-free compression from a fairly high local supersonic Mach number of the order of 1·5 with the more familiar situation on another aerofoil, of the same thickness and at the same conditions, but for which the much more familiar strong pressure rise is present, that is, the strong pressure rise associated with a stationary shock terminating the local supersonic flow. This aerofoil was also carrying quite a lot of lift and the shock-free conditions were obtained on a section designed specifically for an efficient swept-back wing. The contrast between these two situations is indeed quite dramatic, and provides clear evidence that very substantial isentropic compressions can be obtained. The peaky aerofoil represented here was derived in a series of iterative experiments by my colleague, Mr. J. Osborne. The actual experimental observations of the pressures locally on the surface (Fig. 2) indicate some scatter from the mean curve in the supersonic compression region and suggest that weak shocks may have been present

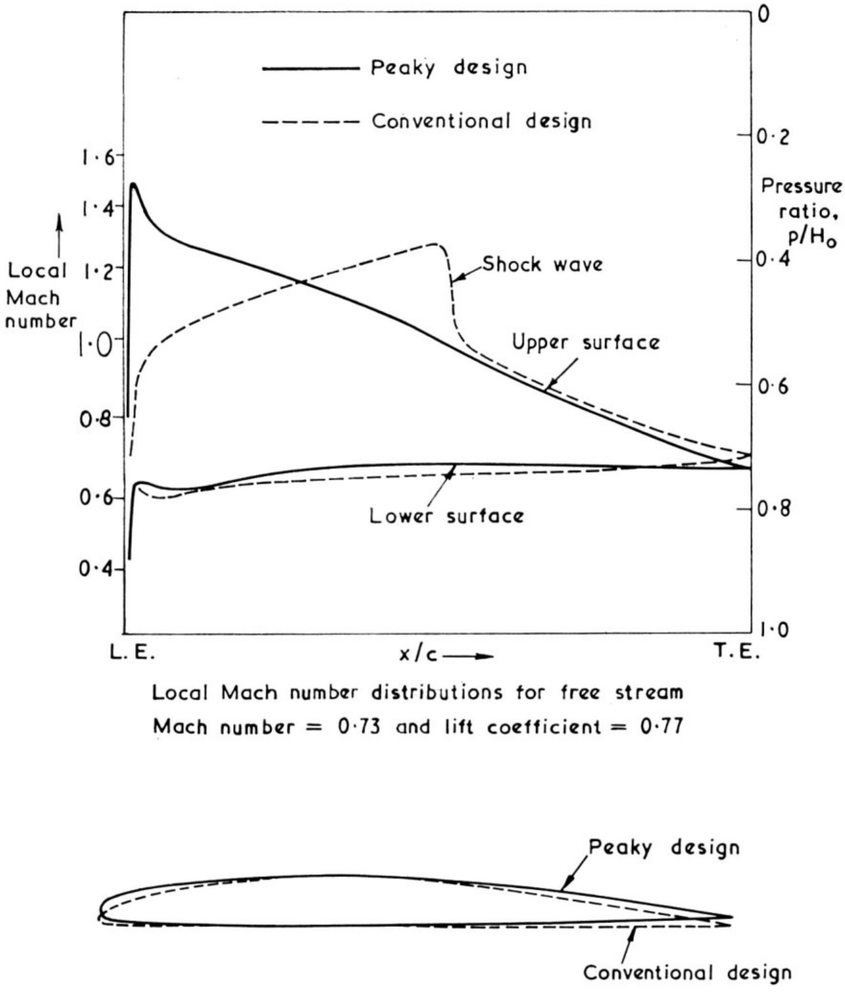


FIG. 1 — Comparisons for a peaky design (N.P.L. 9283) and a good conventional design (N.P.L. 9210); local Mach number distributions and section shapes

in this case; but this does not detract from the achievement in the engineering sense.

The variations of drag coefficient (Fig. 3) with Mach number for the two cases that I have illustrated show quite definitely that for the peaky aerofoil, at the condition for which we saw that the flow was retarded smoothly from a local Mach number of around 1·5, the drag coefficient was no higher than it was at a significantly lower Mach number when there was no local supersonic flow. In fact, if anything it was lower. This is clear evidence that any

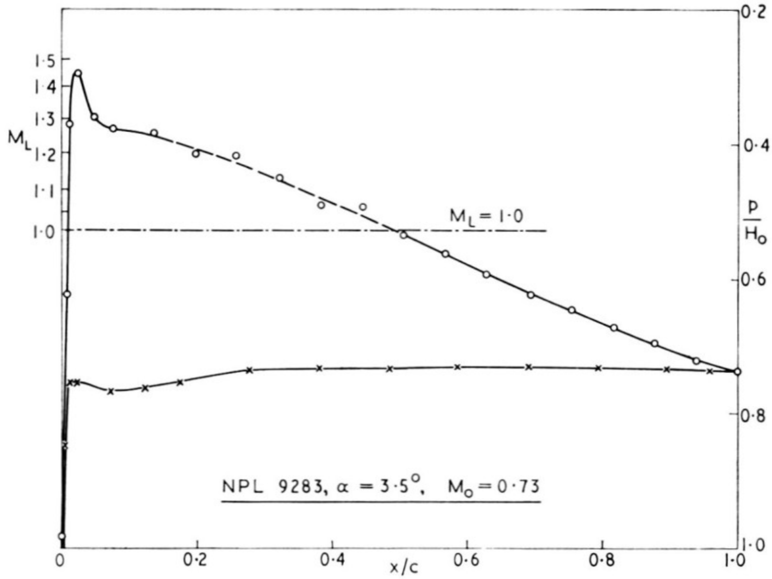


FIG. 2 — Experimental observations for a peaky design with essentially shock-free flow (N.P.L. 9283)

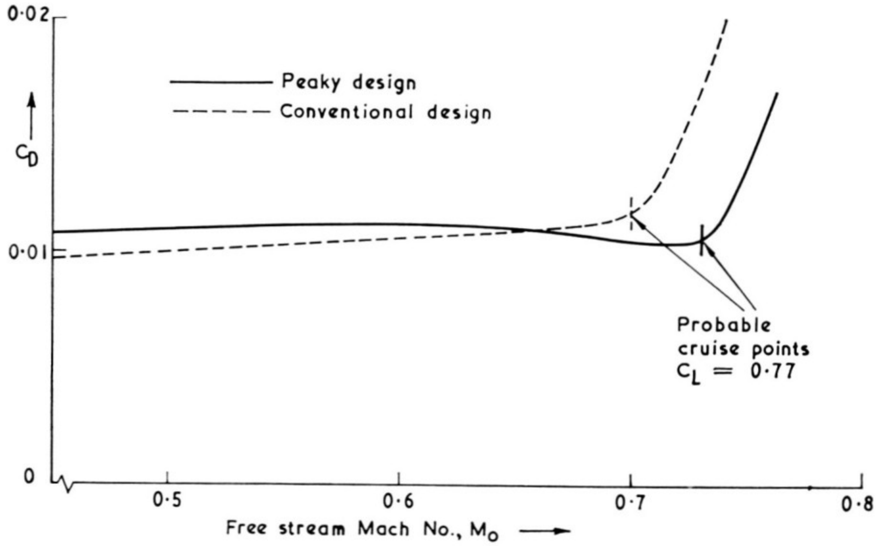
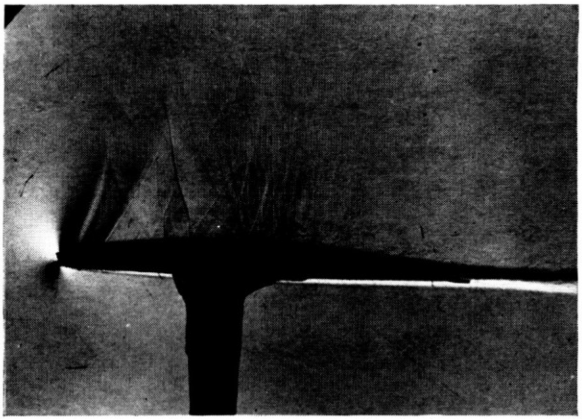


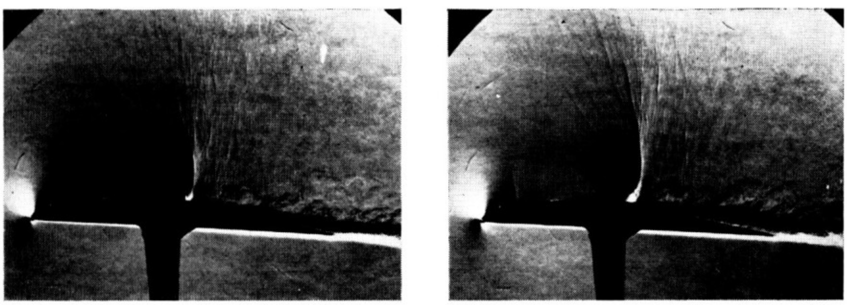
FIG. 3 — Variation of drag coefficient with Mach number at constant incidence for the aerofoils of Fig. 1

shock waves that did exist in the flow were quite insignificant from a practical point of view. On the other hand, the variation of drag coefficient for the conventional type of aerofoil, for which we saw that there were strong stationary shocks, the drag rose quite steeply as soon as these shock waves formed. Eventually, of course, similar shocks form on the peaky aerofoil and cause a similar steep rise in drag, but the advantage in Mach number for a given value of drag is retained.

The comparison between the flow on the peaky aerofoil and the more conventional situation is shown clearly by schlieren photographs (Fig. 4) taken under comparable conditions of free-stream Mach number and lift coefficient.



(*a*) Peaky design; $M_0 = 0{\cdot}73$, $C_L = 0{\cdot}77$



$M_0 = 0{\cdot}72$, $C_L = 0{\cdot}78$ $M_0 = 0{\cdot}74$, $C_L = 0{\cdot}76$

(*b*) Conventional design

FIG. 4 — Schlieren photographs appropriate to the local Mach number distributions shown in Fig. 1

Looking first at the photographs for the more conventional situation, the thing which stands out and dominates the whole of this flow pattern is the strong stationary shock wave. There are other types of weak waves showing on the photographs that I think still have to be considered in finally settling whether completely shock-free flows are obtained in the strict mathematical sense. It should be noted that these photographs were obtained with a very short duration spark exposure that arrests disturbances moving from down-stream, against the stream; the disturbances, of course, form sharp pressure fronts at these speeds. Indeed, for the conventional aerofoil in which the stationary shock is so prominently seen, one can visualise these moving waves building up into the stationary shock and also visualise them moving around the outside of the local patch of supersonic flow. It may well be that these moving waves play quite an essential part in the whole story and they are being studied in detail by my colleague Mr. Moulden at the N.P.L.[1], as well as by Mr. Nieuwland's colleagues at N.L.R.

In the photograph for the case where we noted (Figs. 1 and 2) an essential shock-free compression from a local Mach number of near 1·5, the strong stationary shock is completely absent. However, there are at least three types of weak shock. First, I think we can identify the same moving waves. Second, there are weak disturbances from carborundum grains placed on the surface to fix boundary-layer transition. Third, near the end of the sonic region, some of the weak disturbances could be of the type predicted by Emmons many years ago[2]. Finally, there is in the flow a weak, inclined wave that we suspect arose because, in fixing the rate of the isentropic compression, we pitched it too near to that of a simple wave which we know will give a convergence of characteristics within the local supersonic regime[3]; this shock is therefore one of the type that will occur if the internal structure of the local supersonic flow violates certain conditions. The existence of weak shocks in this case is therefore no indication that they must always occur.

Indeed, there are other examples, of which I can show one (Fig. 5), in which the shock-free local supersonic flow developed quite smoothly as Mach number was increased from the point at which the flow first became super-sonic locally. It can be seen that the pressure rises quite smoothly through the value corresponding to sonic flow. The impression created by this example is that the shock-free flows develop in quite a stable manner over a significant range of free-stream Mach number. This same impression is strengthened by examining the situation for the same 14% thick aerofoil as incidence is increased from a low value (Fig. 6). At the lowest incidence, the occurrence of local supersonic flow leads immediately to the fairly conventional stationary shock. As incidence is increased, however, the shock-free flows develop and the range of Mach number for which these appear to be stable increases.

This aerofoil was designed specifically to achieve this type of flow using the knowledge that we had gained empirically about how to shape the leading

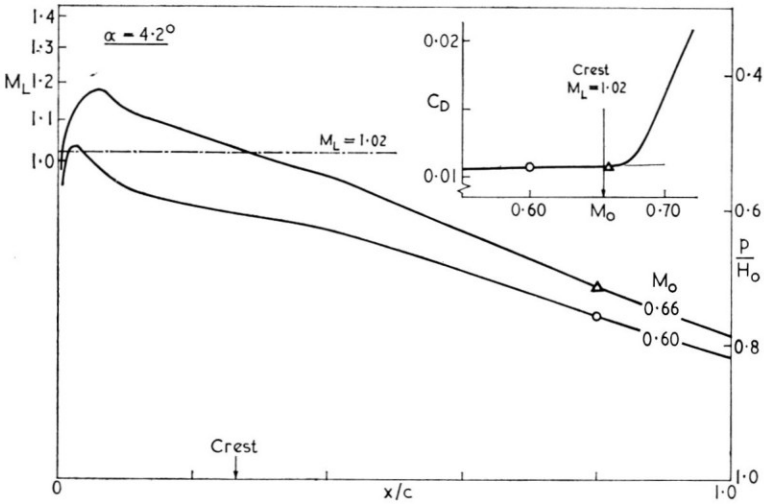


FIG. 5 — Further example of peaky aerofoil design

edge. If this is not done, quite the opposite situation arises in which the flow with strong stationary shock waves seems to be the order of the day (left-hand side of Fig. 6). The differences in the way in which drag varies with Mach number for these various cases demonstrate that these shock waves are not just our interpretation of a pressure jump in the surface pressure, but indeed do give rise to wave drag as one would expect, and this is in contrast to the situation for the aerofoil in which shock-free flow was achieved and for which there were substantial regions of local supersonic flow without any increase in drag.

In conclusion, I would like to stress very strongly the practical importance, not only of the shock-free type of flow that Mr. Nieuwland has demonstrated mathematically and which I have been illustrating from experimental results, but also the wider implications of being able to obtain isentropic compressions from high local Mach numbers in the more general sense, even if at the end of these compressions we accept some weak stationary shock waves. I stress this because the aircraft designer, particularly for the swept wings so frequently used for high subsonic transports and variable-sweep wings — to quote just two examples — is pushing all the time to increase thickness (to reduce structure weight), to carry higher wing-loadings (for efficient cruise), and to keep his sweepback as low as possible (to achieve acceptable take-off and landing conditions and, again, low structure weight). All of these trends tend to increase the local super velocities and hence to increase the pressure on accepting some degree of local supersonic flow. Furthermore, whatever type of flow he chooses at the design condition, he also has to cope with off-design

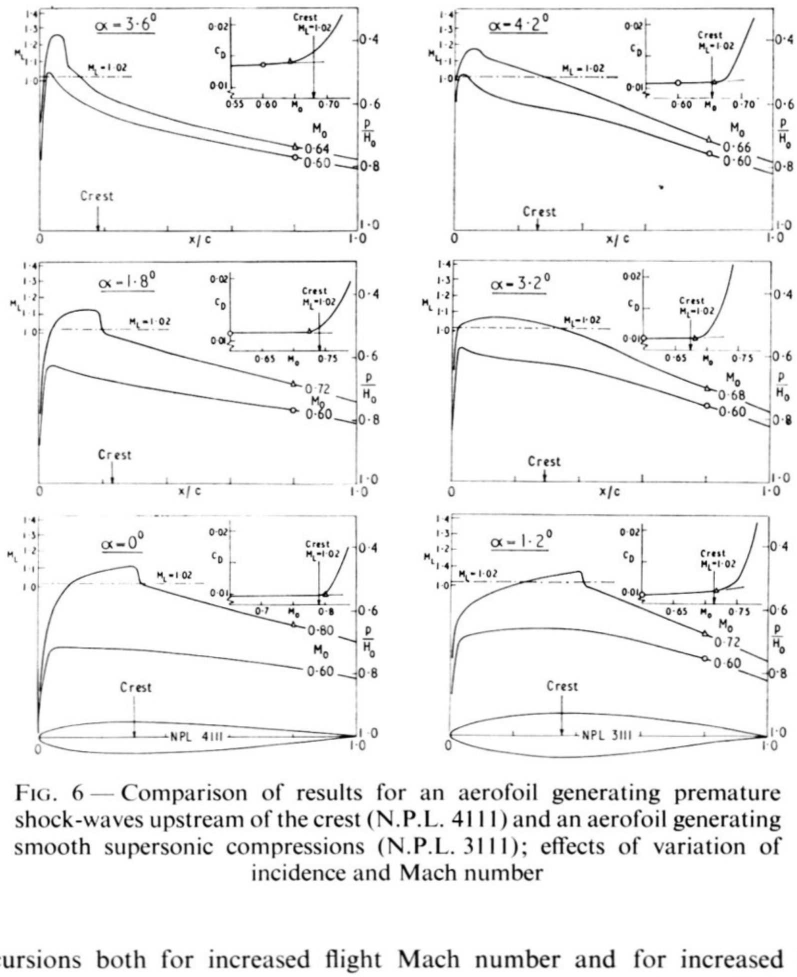


FIG. 6 — Comparison of results for an aerofoil generating premature shock-waves upstream of the crest (N.P.L. 4111) and an aerofoil generating smooth supersonic compressions (N.P.L. 3111); effects of variation of incidence and Mach number

excursions both for increased flight Mach number and for increased C_L in gust-encounters and manoeuvres. These off-design excursions will inevitably lead to the development of local supersonic flows over the surface of his wings. Our task therefore is to ensure that these local supersonic flows can develop smoothly and that the deceleration from the high velocities can occur either without shock waves or with shock waves that are 'little ones' that can be tamed.

REFERENCES

(1) MOULDEN, T. H., COX, I. J., STRINGFELLOW, V. A., A preliminary experimental investigation of shock-wave development on aerofoils. N.P.L. Aero Note 1042 (1966)

(2) EMMONS, H. W. The theoretical flow of a frictionless adiabatic, perfect gas inside a two-dimensional hyperbolic nozzle. NACA TN 1003 (1946)
(3) MOULDEN, T. H. Some comments on the conditions in a local supersonic flow region. N.P.L. Aero Report 1160 (1966)

R. C. Lock (National Physical Laboratory, Teddington, Middx., England): (*written comment — not delivered at lecture*); I should like to amplify Mr. Pearcey's remark on the importance of this work in providing a set of standard 'exact' solutions by which other numerical methods can be judged. I am referring here mainly to methods for calculating the flow about a given aerofoil shape. These are of two kinds — those based on finite difference techniques which aim to provide accurate numerical solutions of the exact equations of motion; and approximate semi-empirical methods whose aim is simply to provide results of adequate 'engineering' accuracy with very much less computational effort than could ever be possible with any more refined method. Either of these approaches is (at present at least) likely to be restricted to examples when the flow is sub-critical (completely subsonic) everywhere or at most has a very small supersonic patch, unlike most of the more advanced examples that Mr. Nieuwland has shown us. I would therefore like to ask him if he would be prepared to make available a number of suitable examples of this kind — covering a range of thickness ratios, maximum thickness position and (eventually) lift coefficient — for the benefit of other workers on this very important problem.

B. M. Spee (Nationaal Lucht- en Ruimtevaartlaboratorium, Amsterdam): Mr. Nieuwland pointed at the desirability to have a criterion for the stability of two-dimensional transonic potential flows. It may be of interest to say a few words more about the unsteady aspects of such flows.

Wind tunnel experiments show that the formation of a steady shock wave in two-dimensional transonic flows is generally preceded by a concentration of unsteady waves. Weak unsteady waves can be observed also in Mr. Pearcey's shock-free flow. Moreover, one of the counter arguments in the discussion on the physical significance of two-dimensional potential transonic flows has been that upstream moving disturbances would be unable to propagate forwards into the finite supersonic region and build up into a shock wave.

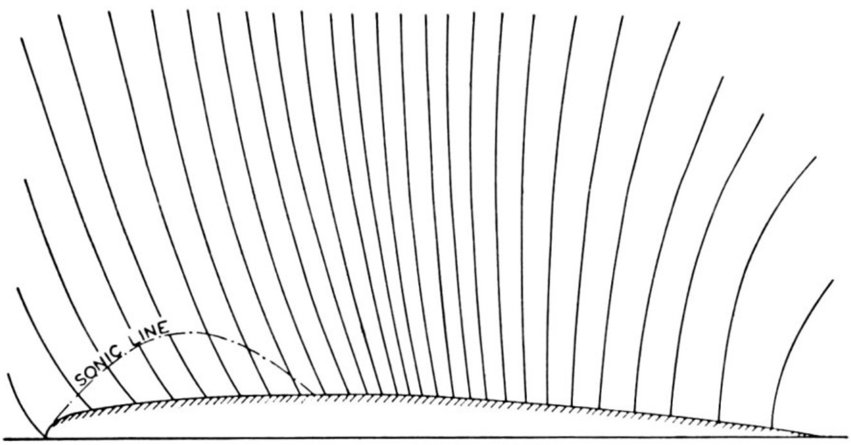
It can be shown that although this argument cannot be put in such a general way, probably a relation exists between the behaviour of disturbance waves and the generation of a shock wave.

Disturbances generated in the subsonic region downstream propagate forwards, and due to the existence in the basic flow field of velocity gradients perpendicular to the chordwise direction, they move faster at a larger distance from the profile than close to the profile. The disturbance waves incline as they move forwards. The inclination depends on the value of the velocity gradients, large gradients give a large inclination, small gradients a small inclination. The waves reach the sonic line at a certain angle with the mean flow and for this reason they can move forwards into the supersonic region.

The inclination of waves reaching the sonic line depends, except on the velocity gradients perpendicular to the chord, on the position of the source. When we restrict ourselves to disturbances generated in the boundary layer and the wake, the inclination at the sonic line is larger for source positions more downstream. However this effect is generally not very large due to the fact that the inclination is mainly defined by the velocity gradients in the neighbourhood of the sonic line where the propagation speed is minimal.

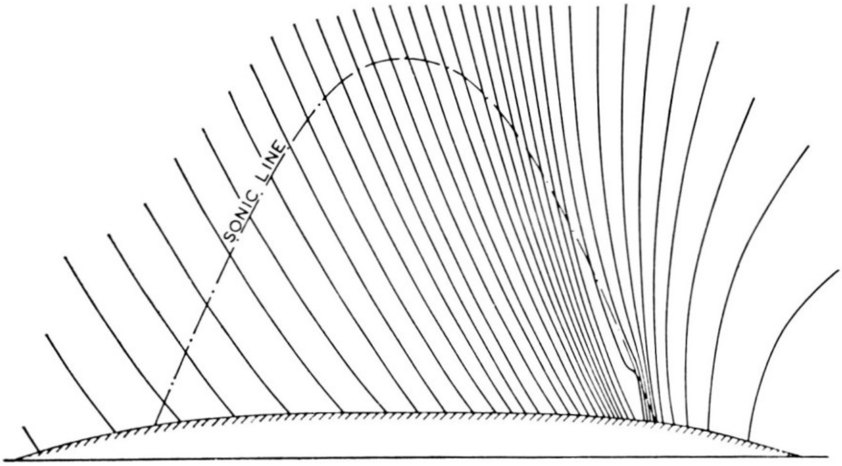
These considerations are illustrated in Figs. 1 and 2. In these figures the phase pattern has been constructed for waves generated by a source located at the trailing edge of two quasi-elliptic aerofoils. The profile given in Fig. 1 has large velocity gradients perpendicular to the chord. Consequently the disturbance waves have a large inclination at the time that they reach the supersonic region and they propagate fast and hardly deformed through this region. The profile given in Fig. 2 has small velocity gradients perpendicular to the chord. The waves have a small inclination at the sonic line and they move slowly into the supersonic region. It appears that they tend to approach characteristic lines of the basic flow field, characteristics being the equilibrium condition for acoustic waves.

It should be noted again that the influence of the position of the source is not very large in general. For the profile of Fig. 1, for instance, all disturbances generated in the boundary layer and the wake, even from rather close behind



AEROFOIL :0.11 - 0.775 - 1.300
CHORD LENGTH :0.33m
SOURCE AT TRAILING EDGE, FREQUENCY - 3kHz

FIG. 1.



AEROFOIL : 0.12-0.700-0.000
CHORD LENGTH : 0.33 m
SOURCE AT TRAILING EDGE, FREQUENCY - 3 kHz

FIG. 2.

the supersonic region, move fast and without difficulty through the supersonic region, their angle being larger than the maximum characteristic angle in the supersonic region. For the profile of Fig. 2 all disturbances generated downstream of the supersonic region behave as shown in this figure. They practically come to a standstill in the downstream part of the supersonic region, and concentrate in a narrow strip. The width of this strip depends on the local velocity gradient in chord direction and diminishes as the velocity gradient increases.

So far it has been assumed that the unsteady waves, which will always be generated in the real transonic flow around a profile in the boundary layer and the wake, do not interact with the basic potential flow. It has been shown however by Kuo[(1)] that waves increase in amplitude when travelling upstream in a decelerating transonic flow region consuming the kinetic energy of the steady flow. Such a region with decelerating flow is the downstream part of the supersonic region.

In a flow with small velocity gradients perpendicular to the chord where the waves concentrate and accumulate in this region the amount of kinetic energy withdrawn from the steady flow becomes large, and this energy consumption takes place in a very small region, one can no longer consider the waves to be superimposed on the potential flow. This possibly indicates the breakdown of the smooth flow and the formation of a shock.

It can be concluded that in order to stay far away from a situation which probably favours the generation of a stable shock-wave the flow has to meet the following requirements: the boundary layer and the wake should be quiet, the velocity gradients perpendicular to the chord should be large and the gradient in chord direction small. The last requirements apply mainly to the region around the downstream part of the sonic line.

It appears consequently that profiles with a peaky pressure distribution are favourable and that a stability criterion based on the behaviour of unsteady waves will correspond to Mr. Pearcey's experience for shock-free flows.

References

(1) Kuo, Y. H. On the stability of two-dimensional smooth transonic flows. J.A.S., **18**, pp. 1–6 (1951).

(2) Spee, B. M. Wave propagation in transonic flow past two-dimensional aerofoils. N.L.R. — TN T. 123 (1966).

Dr. K. Kraemer (A.V.A., Bunsenstr. 10, 34 Goettingen, Germany): I want to call attention to experiments at the Max-Planck-Institut fur Stromungsforschung at Goettingen begun by Dr. Koppe and presently continued by dipl. phys. Meyer. In a two-dimensional channel a subsonic Chaplygin flow with enclosed supersonic region was set up. Optical survey revealed the flow to be 'shock-free', and to be rather insensitive to slight perturbations of the upstream conditions. A shock could, of course, be produced by a sufficient increase in channel-massflow. The boundary layer at the curved channel walls was sucked off, but that at the plane side walls could not be controlled. Therefore the intepretation of results in terms of two-dimensional flow was difficult. I feel, however, that the explanation just given by Mr. Spee is most convincing.

Evidence that a shock must occur at free stream Mach number exactly equal to one follows from an axisymmetric, self similar solution of the transonic equations of motion obtained by E. A. Muller and K. Matschat, Goettingen (Congress at *Stresa*). This solution is analogous to the familiar incompressible dipole-flow and should apply to the far field of any body at $M = 1$. It does contain a shock.